Golub-Kahan iterative bidiagonalization and stopping criteria in ill-posed problems

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1 Introduction

Consider estimating \( x \) from the system of ill-posed linear algebraic equations

\[
Ax \approx b, \quad b = b^{\text{exact}} + b^{\text{noise}}, \quad A \in \mathbb{R}^{n \times m}, \quad b \in \mathbb{R}^n,
\]

where the uninteresting case is excluded by the assumption \( A^Tb \neq 0 \).

Linear approximation problems of this form arise in a broad class of scientific and technical disciplines and applications – medical image deblurring (tomography), bioelectrical inversion problems, geophysics (seismology, radar or sonar imaging), astronomical observations. Here the noise component \( b^{\text{noise}} \) in \( b \) is usually unknown and \( \| b^{\text{noise}} \| \ll \| b^{\text{exact}} \| \), where \( \| . \| \) represents the standard Euclidean norm. The matrix \( A \) is ill-conditioned and a small perturbation on \( b \) typically causes large changes in the estimated solution \( x \). Moreover, the matrix \( A \) is often numerically rank deficient and/or it has small singular values, but without a well defined numerical rank. In such cases (total) least squares techniques \([3, 8]\) might give a solution that is absolutely meaningless, because it is dominated by errors present in the data and possibly also by computational errors. Regularization techniques based on Golub-Kahan bidiagonalizations \([5]\) are often used to suppress the effect of errors and extract the essential information about the system, see, e.g., LSQR \([10]\) and hybrid methods \([1, 2, 6, 9]\).

2 Golub-Kahan bidiagonalization

For \([b, A]\) large consider the partial lower Golub-Kahan iterative bidiagonalization in the following form. Given the initial vectors \( v_0 \equiv 0, u_1 \equiv b/\beta_1 \), where \( \beta_1 \equiv \| b \| \neq 0 \), the algorithm computes for \( i = 1, 2, \ldots \)

\[
\begin{align*}
\alpha_i v_i &= A^T u_i - \beta_i v_{i-1}, \quad \| v_i \| = 1, \\
\beta_{i+1} u_{i+1} &= A v_i - \alpha_i u_i, \quad \| u_{i+1} \| = 1
\end{align*}
\]

until \( \alpha_i = 0 \) or \( \beta_{i+1} = 0 \), or until \( i = \min\{n, m\} \). Denote \( U_k \equiv (u_1, \ldots, u_k), V_k \equiv (v_1, \ldots, v_k) \) the resulting matrices with orthonormal columns, and

\[
L_k \equiv \begin{pmatrix}
\alpha_1 \\
\beta_2 & \alpha_2 \\
\varepsilon_L & \ldots & \ldots \\
\beta_k & \alpha_k
\end{pmatrix}, \quad L_{k+} \equiv \begin{pmatrix}
L_k \\
\beta_{k+1} e_k^T
\end{pmatrix}.
\]
In hybrid methods, the problem (1) is first projected onto a lower dimensional subspace using the bidiagonalization algorithm (2), and then some type of inner regularization is used to solve the projected problem

\[ L_{k+} y \approx \beta_1 e_1 \]  

(3)

with the lower bidiagonal matrix \( L_{k+} \). The bidiagonalization is stopped when the solution of (3) is acceptable. The stopping criterion is based, e.g., on the estimation of the L-curve using the L-ribbon [4], the discrepancy principle and generalized cross validation [1, 2]. Similar techniques has been thoroughly studied and compared, e.g., in [6].

Recently, it has been proved that the Golub-Kahan bidiagonalization leads to a fundamental decomposition of data, revealing the so called core problem, see [11]. The core problem theory shows that in exact arithmetic the necessary and sufficient information from (1) is extracted to (3) as soon as a zero value is encountered either on the diagonal or on the superdiagonal of the matrix \( L_{k+} \). This forms an important theoretical background for hybrid methods. At each step \( k \) hybrid methods compute the system that is an approximation to the core problem and thus the approximate solution cannot contain redundant or irrelevant information.

Stopping the bidiagonalization process when a zero value is encountered is numerically not suitable and one should ask how to numerically indicate the separation of the core problem. Then the question arises whether and when the main submatrix of the bidiagonal matrix \( L_{k+} \) can be considered a sufficiently good approximation to the whole core problem matrix.

3 Outline

The application of presented facts to ill-posed problems has been studied by D. Sima and S. Van Huffel [12], and by P. C. Hansen, M. E. Kilmer and R. Kjeldsen [7]. In this contribution we investigate a possibility of direct using of the information from the bidiagonalization for constructing an effective stopping criteria in solving ill-posed problems by hybrid methods. Especially we show, how the information about the level of the noise in \( b \) can be revealed from the vectors \( u_k \) generated by (2).

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References


